Exercise | We compute the bound states of the finite square well in one dimension.

(Source: Griffiths QM, Section 2.6)

The finite square well is given by the potential

$$V(x) = \begin{cases} -V_0 , & |x| < a \\ 0 , & |x| > a \end{cases}$$

where $V_0 > 0$ and $E < 0$ (as we are interested in bound states.

We solve the 1D Schrödinger equation in each region.

Case 1: $x < -a$

$$-\frac{\hbar^2}{2m}\psi'' = E\psi \implies \psi'' = -\frac{2mE}{\hbar^2}\psi = \kappa^2\psi$$

$$\implies \psi = Ae^{\kappa x} + Be^{-\kappa x}.$$

We only care about normalizable solutions, so we're left with

$$\psi = Ae^{\kappa x}.$$

Case 2: $x > a$

By the same arguments as above, and the symmetry of the potential about $x=0$, we have

$$\psi = Ae^{-\kappa x}.$$

Case 3: $|x| < a$.

$$-\frac{\hbar^2}{2m}\psi'' - V_0\psi = E\psi \Rightarrow \psi'' = -\frac{2m(E+V_0)}{\hbar^2}\psi.$$

In order for ψ to be normalizable, we require $-V_0 < E$, but also to get bound states, we require $E < 0 \Rightarrow -V_0 < E < 0 \Rightarrow 0 < E+V_0 \Rightarrow$

$\psi'' = -l^2\psi \Rightarrow \psi = B\cos lx + C\sin lx$. As we

know the solutions will be symmetric under $x \mapsto -x$, we have

$$\psi = B\cos lx.$$

We now match boundary conditions. Since the potential is nowhere infinite, we know the wavefunction and its derivatives will be continuous everywhere. Thus, our boundary conditions gives us

$$Ae^{-\kappa a} = B\cos la \,, \quad -\kappa Ae^{-\kappa a} = -lB\sin la \Rightarrow$$

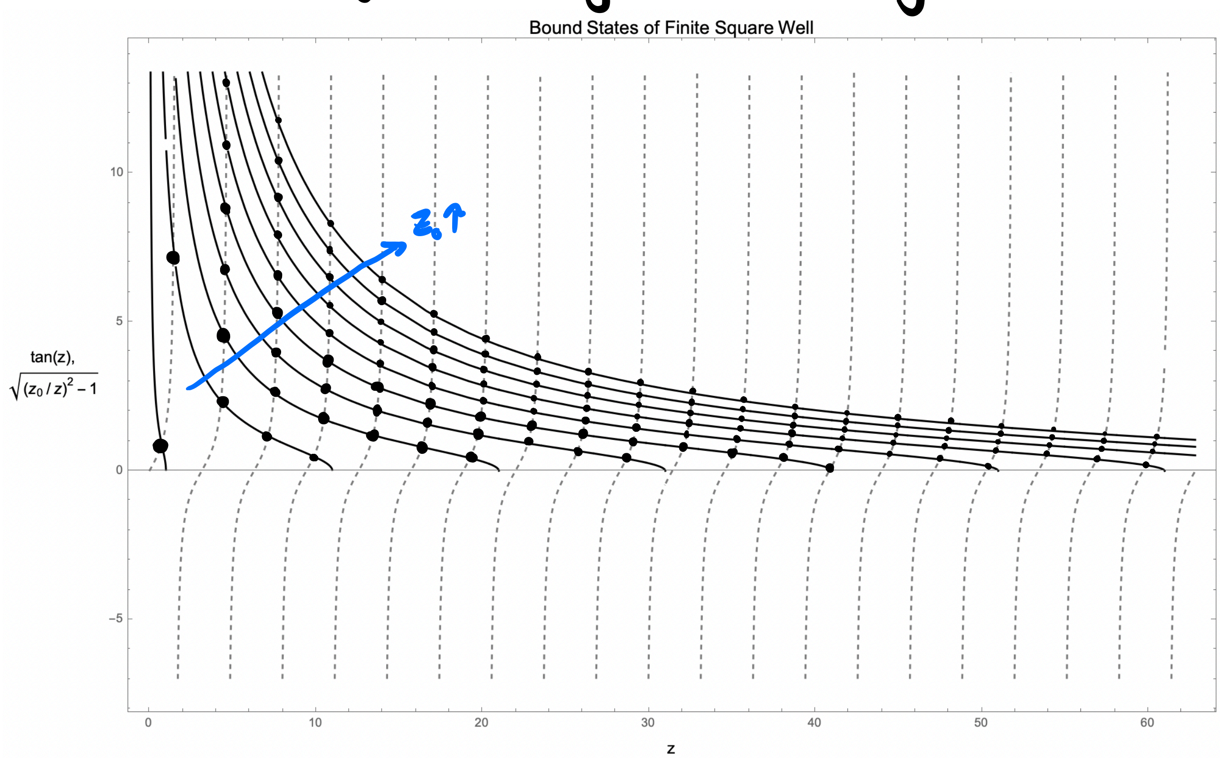
$$\kappa = l\tan la \Rightarrow \kappa a = la\tan la.$$

Now, let $z = la$, $z_0 = \frac{a}{\hbar}\sqrt{2mV_0}$. Then,

$$a^2(\kappa^2 + l^2) = -\frac{2mEa^2}{\hbar^2} + \frac{2m(E+V_0)a^2}{\hbar^2} = z_0^2 \Rightarrow$$

$$\kappa a = \sqrt{z_0^2 - z^2} \Rightarrow \boxed{\tan z = \sqrt{z_0^2/z^2 - 1}}$$

The bound state energies of this system will satisfy this transcendental equation.



Clearly, as we increase z_0, the bound states approach the vertical asymptotes of the tangent function: $z = n\pi/2$ (n odd) $\longrightarrow$

$$\frac{n\pi}{2} = a\sqrt{\frac{2m(E+V_0)}{\hbar^2}} \Rightarrow \boxed{E_n = \frac{\hbar^2\pi^2 n^2}{8ma^2} - V_0, \; n \in \text{odds}}$$

for wide and deep wells. This is a shifted version of the infinite square well spectrum for a box of size $2a$.

At the other extreme, no matter how small we make z_0, it will still intersect the tangent at one point, since

$$\tan z \approx z, \quad \text{and} \quad \sqrt{(z_0/z)^2 - 1} \quad \text{has a vertical asymptote at } z=0$$
and vanishes at $z = z_0$.

Both curves being continuous implies that they must intersect somewhere between $z=0$ and $z = z_0$, regardless how small z_0 gets. (Here, I am assuming $z < \pi/2$.)

This implies that the finite square well will always have a bound state in one dimension, even if it is very thin or very shallow.